Determination of pure spin state from three measurements

1993 J. Phys. A: Math. Gen. 26 L579

(http://iopscience.iop.org/0305-4470/26/13/005)

J. Phys. A: Math. Gen. 26 (1993) L579–L582. Printed in the UK

LETTER TO THE EDITOR

Determination of pure spin state from three measurements

I D Ivanovic

Faculty of Physics, University of Belgrade, POB 550, 11000 Belgrade, Yugoslavia

Received 24 February 1993

Abstract. We show that the state determination of a pure *spin state* can be obtained from the results of three Stern–Gerlach type measurements. If the initial state is $m=0$ state, the selection of measurements cannot be made in advance.

The problem of state determination of a pure state in spin space has recently been discussed in several papers, e.g. [1, 2]. The aim of this letter is to consider determination of a pure *spin state* which is a relatively simple task, when compared to the determination of an arbitrary pure state [1, 2] or an arbitrary mixed state [3, 4] in spin space. We show that state determination procedure for a pure spin state consists of three spin component measurements. The first measurement is an arbitrary one (concerning the orientation) while the following ones must be chosen on the basis of results of previous ones. A prearranged set of measurements may fail to give sufficient data for a state determination.

To start with, by a pure spin state for spin j we will assume an eigenstate of operator $J_n = \boldsymbol{n}\cdot\boldsymbol{J} = n_xJ_x + n_yJ_y + n_zJ_z$ where $|\boldsymbol{n}|=1$, in a complex $(2j+1)$-dimensional space. Accordingly, any state can be labelled by the orientation $\boldsymbol{n}$ and the eigenvalue of J_n, m; $J_n|\boldsymbol{n}, m\rangle = m|\boldsymbol{n}, m\rangle$ where $-j \leqslant m \leqslant j$. Starting from some basis set of vectors, e.g. the set of eigenvectors of J_z, $|z, m\rangle$ all other spin pure states can be obtained by applying rotation matrix $\boldsymbol{D}^{(j)}(\alpha, \beta, \gamma)$ to the chosen set. The rest of the pure states are non-spin states and their interpretation is given in [5].

We assume an ideal state determination: the ensemble prepared in an unknown pure spin state is available in a sufficient number of identical replicas. On each replica a measurement by means of a standard Stern–Gerlach measurement is performed and by the measurement result we assume the probability distribution obtained from the measurement

$$p_j, p_{j-1}, \ldots p_{-j}.$$

Also, the measurement result $\{p_m\}_{m=j}^{m=-j}$ must coincide with probabilities obtained from calculations.

The determination is completed when one is able to determine the orientation of the initial preparation $\boldsymbol{n}_0$ and the eigenvalue of $J_{\boldsymbol{n}_0}$, m_0.

Let $\boldsymbol{n}_1$ denote the orientation of the first measurement performed, that of J_1, and let $\{p_m^{(1)}\}_{m=j}^{m=-j}$ be the result of the measurement. Obviously, $p_m^{(1)} = (d_{m_0,m}^{(j)}(\beta_1))^2$ where $d_{m_0,m}^{(j)}(\beta_1) = [\boldsymbol{D}^{(j)}(0, \beta_1, 0)]_{m_0,m}$ and $\beta_1 = \angle(\boldsymbol{n}_0, \boldsymbol{n}_1)$. In particular [6]

$$p_j^{(1)} = (d_{m_0,j})^2 = \binom{2j}{j-m_0}(\cos(\beta_1/2))^{2(j+m_0)}(\sin(\beta_1/2))^{2(j-m_0)} \tag{1}$$

and

$$p_{-j}^{(1)}=(d_{m_0,-j})^2=\binom{2j}{j-m_0}(\cos(\beta_1/2))^{2(j-m_0)}(\sin(\beta_1/2))^{2(j+m_0)}. \tag{2}$$

Using the fact that

$$\langle J_1\rangle = m_0\cos(\beta_1) \tag{3}$$

from (1)–(3), when $\langle J_1\rangle\neq 0$, one obtains an equation for m_0

$$\left(\frac{\langle J_1\rangle^2}{m_0^2-\langle J_1\rangle^2}\right)^{m_0}=\frac{p_j^{(1)}}{p_{-j}^{(1)}} \tag{4}$$

and an equation for β_1

$$\cot^2(\beta_1)=\frac{\langle J_1\rangle^2}{m_0^2} \tag{5}$$

where (4) and (5) are valid if $m_0\neq 0$.

As a consequence, from a single measurement (if $\langle J_1\rangle\neq 0$) one may infer the value of m_0 and β_1. The knowledge of β_1 reduces the set of allowed orientations for $\boldsymbol{n}_0$ to a cone $\mathrm{Con}(\boldsymbol{n}_1,\beta_1)$ with the apex at the origin, $\boldsymbol{n}_1$ as the axis and β_1 as the half-angle. Reorientation of an axis (i.e. change of $J_i\to -J_i$) allows one to consider only positive m_0 and consequently only the 'upper' parts of the cones in question. This 'upper' part of the cone we denote by $\mathrm{Con}^+(\boldsymbol{n}_1,\beta_1)$.

From the next measurement, e.g. along $\boldsymbol{n}_2$, one obtains β_2 and $\boldsymbol{n}_0\in(\mathrm{Con}^+(\boldsymbol{n}_1,\beta_1)\cap\mathrm{Con}^+(\boldsymbol{n}_2,\beta_2))$. The apex of all cones we are dealing with is at the origin. Then one should choose between, at most two, remaining orientations for $\boldsymbol{n}_0$ and this can be made after the third measurement made along any $\boldsymbol{n}_3$ which does not belong to the plane defined by $(\boldsymbol{n}_1,\boldsymbol{n}_2)$.

A slightly different approach must be made for integer j in the case when $\langle J_1\rangle=0$. Such a result may occur for two reasons only: either the initial state is an $m_0=0$ state with an arbitrary $\boldsymbol{n}_0$ or the initial state's m_0 is arbitrary but $\angle(\boldsymbol{n}_0,\boldsymbol{n}_1)=\pi/2$. In both cases the probability distribution is a symmetrical one, i.e. $p_m=p_{-m}$.

What follows is the proof that, except for ($j=1$), it is possible to differentiate between a probability distribution $\{p_{0,m}(\beta)\}$ obtained from some $|\boldsymbol{n}_0,0\rangle$ and a probability distribution $\{p_{m_0,m}(\pi/2)\}$ from some $|\boldsymbol{n}_0,m_0\rangle$ when $\angle(\boldsymbol{n}_0,\boldsymbol{n}_1)=\pi/2$. In fact we will show that if two first probabilities are equal, i.e.

$$p_{0,j}(\beta)=p_{m_0,j}(\pi/2)\qquad p_{0,j-1}(\beta)=p_{m_0,j-1}(\pi/2) \tag{6}$$

then the third pair must be different, i.e. $p_{0,j-2}(\beta)\neq p_{m_0,j-2}(\pi/2)$.

The following equations are a consequence of (6):

$$p_{m_0,j-1}(\pi/2)=(2m_0^2/j)\,p_{m_0,j}\qquad \cot^2(\beta)=\frac{m_0^2}{j^2}. \tag{7}$$

Probabilities $p_{0,j-2}(\beta)$ and $p_{m_0,j-2}(\pi/2)$ may be obtained from the recurrence relation [7]

$$\sqrt{(j-m)(j+m+1)}\,\sin(\beta)d_{m_0,m+1}(\beta)+\sqrt{(j+m)(j-m+1)}\,\sin(\beta)d_{m_0,m-1}(\beta)$$
$$=2(m\cos(\beta)-m_0)d_{m_0m}(\beta). \tag{8}$$

Introduction of the particular values in (8), rearrangement, squaring and inserting the values from (7) one obtains that $p_{0,j-2}(\beta)=p_{m_0,j-2}(\pi/2)$ if, and only if, the next four equations can be solved in m_0:

$$m_0^4 \pm m_0^2 j = j^2(j-1)(j-1\pm 1). \qquad (9)$$

It is easy to check that (9) has no solutions satisfying $-j \leqslant m_0 \leqslant j$ except for $j=1$ when the solution is $m_0=\pm 1$. The solution for the $j=1$ case will be left until the conclusion of this letter while for all other cases it is possible to infer value of m_0 for the initial state.

Determination of $\boldsymbol{n}_0$ for an $m_0=0$ state is slightly different from the determination of an $m_0 \neq 0$ state due to the fact that reorientation of axes does not simplify the problem and we will show that for every prearranged set of three measurements (and orientations) at least a pair of pure spin states exists which cannot be differentiated by the chosen set of three measurements.

For a given axis, e.g. $\boldsymbol{n}_1$ and angle β_1, obtained from a measurement assuming that the initial state is an $m_0=0$ pure spin state admissible $\boldsymbol{n}_0$ should lie on a proper cone $\mathrm{Con}(\boldsymbol{n}_0, \beta)$. After the second measurement, if it is chosen before the result of the first measurement is known it may occur that $\mathrm{Con}(\boldsymbol{n}_1, \beta_1) \cap \mathrm{Con}(\boldsymbol{n}_2, \beta_2)$ contains 4 admissible orientations for the unknown $\boldsymbol{n}_0$. An unfortunate *a priori* choice of third measurement may discard a pair of orientations but may be unable to decide between the remaining two orientations. Such a case occurs, for example, if one tries to detemine an $m_0=0$ state from the results of measurements of J_x, J_y, J_z; when four orientations may remain.

We will now show how to find a pair of states which cannot be differentiated by a chosen set of three measurements. Instead of solving the set of equations we will give a proof using only elementary geometry.

Consider a unit sphere in three-dimensional space and it's projection into the plane defined by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ (figure 1). Projection of the intersections of a cone with the sphere is represented by two parallel lines equidistant from the centre of the projection. Projections of cone intersections are orthogonal to the corresponding axis. Intersection

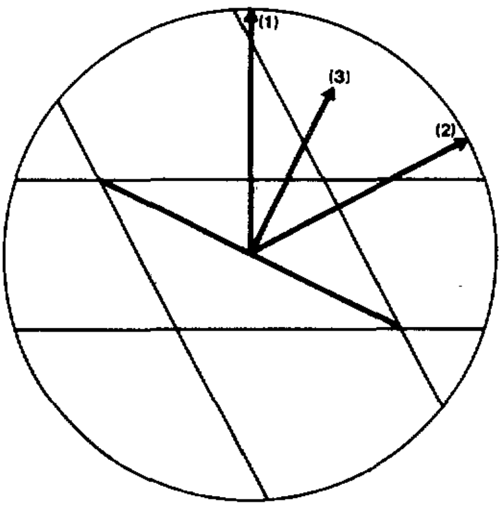

Figure 1. Projection of three predetermined axes and the corresponding pair of pure states giving identical probability distributions.

between two such pairs of parallel lines makes a rhomboid. Now, every diagonal of the rhomboid is a projection of two admissible orientations for $\boldsymbol{n}_0$ being the intersection of two cones. If third measurement should fail in a complete state determination it's projection must be orthogonal to one of the rhomboid's diagonals. In that case one can construct a cone around $\boldsymbol{n}_3$ which will intersect two first cones along two lines which will be possible orientations for $\boldsymbol{n}_0$.

The proof is now obvious; given three orientations one should draw a line orthogonal to the projection of the third orientation through the origin. This line is then one of the diagonals of the rhomboid one still has to construct. Choosing different values for β_1 and β_2 it is always possible to construct a family of desired rhomboids. In the case when projection of the third orientation lies along one of two first orientations the rhomboid will reduce to a line while the corresponding angle is $\pi/2$.

This also gives the answer to the question: 'How to perform a pure spin state determination from three measurements for $m_0=0$ state?' The answer is: 'Next measurement should be chosen on the basis of the results of the previous measurement.' The following is an example. If the first measurement along $\boldsymbol{n}_1$ resulted in β_1 the second measurement should be made along $\boldsymbol{n}_2$ such that $\angle(\boldsymbol{n}_1, \boldsymbol{n}_2) = \pi/2 - \beta_1$. This ascertains only two possible orientations for $\boldsymbol{n}_0$. The third measurement should be made along one of the possible orientations for $\boldsymbol{n}_0$ which will accomplish state determination.

Instead of a conclusion we will return to the $j=1$ case. The probability distribution which hides the nature of the initial state is $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ i.e. either $\beta=\pi/2$ and $m_0=\pm1$ or $\beta=\pi/4$ and $m_0=0$. The second measurement should be made along $\boldsymbol{n}_2$, $\angle(\boldsymbol{n}_1, \boldsymbol{n}_2) = \pi/4$. If the probability distribution is again the same one, the third measurement should be made along $\boldsymbol{n}_3 \angle(\boldsymbol{n}_3, \boldsymbol{n}_1) = \pi/4$, $\angle(\boldsymbol{n}_3, \boldsymbol{n}_2) = \pi/4$ which will fulfill the desired task.

Finally, it is obvious that Stern–Gerlach type of measurements are redundant for the determination of $m_0 \neq 0$ states; only mean values of J_i are necessary. For an $m_0=0$ state, again one does not need a complete probability distribution; only three probabilities from the first measurement are necessary to recognize it's nature and from other probability distributions one needs two probabilities $p_{\pm j}(\beta_2)$ and $p_{\pm j}(\beta_3)$ obtained from properly chosen measurements.

References

[1] Bohn J 1991 *Phys. Rev. Lett.* **66** 1547
[2] Weigert S 1992 *Phys. Rev.* A **45** 7688
[3] Ivanovic I D 1981 *J. Phys. A: Math. Gen.* **14** 3241
[4] Ivanovic I D 1983 *J. Math. Phys.* **24** 1199
[5] Swift A R and Wright R 1980 *J. Math. Phys.* **21** 77
[6] Vigner E P 1959 *Group Theory* (New York: Academic Press) ch 15, eq (15.27)
[7] Louck J D and Biedenharn L C 1981 *Angular Momentum in Quantum Physics* (Reading, MA: Addison-Wesley) vol 1, ch 3, eq (3.84)